

Management Science Seminars

An Introduction to Linear Programming: The RMC Case

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1 Introduction to the Case Study

Every production manager faces a fundamental allocation question: how should a firm divide scarce resources among competing products to earn the highest profit? When both the objective and the resource limits are linear functions of the decision quantities, this allocation problem is a **linear program (LP)**. We develop the core ideas of linear programming—model formulation, graphical solution, algebraic standard form, and matrix representation—through a concrete blending problem at RMC, Inc.

1.1 Case Description: RMC, Inc.

RMC, Inc. is a chemical manufacturer that blends three raw materials (Materials I, II, and III) to produce two commercial products: a **fuel additive** and a **solvent base**.

Each unit of output requires a fixed mix of raw materials:

- Producing 1 unit of fuel additive requires 1 ton of Material I, 2 tons of Material II, and 1 ton of Material III.
- Producing 1 unit of solvent base requires 0 tons of Material I, 3 tons of Material II, and 1 ton of Material III.

After direct variable costs are deducted, the unit profit contributions are:

- \$5 per unit of fuel additive produced,
- \$7 per unit of solvent base produced.

For the upcoming planning period, RMC's raw material inventory is fixed at the following capacities:

- 6 tons of Material I,
- 19 tons of Material II,
- 8 tons of Material III.

Management wants to determine how many units of fuel additive and solvent base to produce to maximize total profit contribution without exceeding the available supply of any raw material.

2 Step 1: Problem Formulation

Before writing equations, we structure the operational problem into four components: decisions, data, objective, and constraints.

1. **Identify the Decision Variables:** What quantities can management directly control?
 - The production quantity of fuel additive, denoted by F .
 - The production quantity of solvent base, denoted by S .
2. **Tabulate the Input Data:** Grouping input-output coefficients, unit profits, and resource capacities into a single table clarifies how the decisions interact.

Table 1: Input-output requirements, resource capacities, and unit profits for RMC, Inc.

Raw Material	Fuel Additive (F)	Solvent Base (S)	Available Capacity (Tons)
Material I	1	0	6
Material II	2	3	19
Material III	1	1	8
Profit per Unit	\$5	\$7	—

3. **State the Objective:** Maximize the total profit contribution earned during the planning period.
4. **State the Constraints:**
 - Total consumption of Material I cannot exceed the 6 tons available.
 - Total consumption of Material II cannot exceed the 19 tons available.
 - Total consumption of Material III cannot exceed the 8 tons available.
 - Production quantities cannot be negative.

3 Step 2: Mathematical Modeling (Translation)

We now translate the verbal formulation in Table 1 into an algebraic model. Define the two decision variables as:

- F : number of units of fuel additive produced,
- S : number of units of solvent base produced.

Each unit of fuel additive contributes \$5 and each unit of solvent base contributes \$7, so the **objective function** for total profit Z (in dollars) gives the optimization objective:

$$\text{Maximize } Z = 5F + 7S.$$

Reading across each row of Table 1 gives one linear inequality per raw material, limiting total usage to available supply:

$$\begin{aligned} \text{Subject to (s.t.): } F &\leq 6 && \text{(Material I),} \\ 2F + 3S &\leq 19 && \text{(Material II),} \\ F + S &\leq 8 && \text{(Material III).} \end{aligned}$$

Because negative production has no physical meaning, we include **non-negativity constraints**:

$$F \geq 0, \quad S \geq 0 \quad (\text{Non-negativity}).$$

4 Step 3: The Graphical Solution Approach

Because the RMC model has only two decision variables, we can solve it geometrically in the (F, S) -plane, with F on the horizontal axis and S on the vertical axis.

1. **Plot the Constraint Boundaries:** Each linear inequality defines a half-space bounded by a straight line:
 - Material I boundary: $F = 6$,
 - Material II boundary: $2F + 3S = 19$,
 - Material III boundary: $F + S = 8$.
2. **Identify the Feasible Region:** The **feasible region** is the set of all points (F, S) that satisfy all three resource inequalities together with $F \geq 0$ and $S \geq 0$ (shaded in Figure 1).
3. **Find the Optimal Solution:** For any fixed profit level Z , the equation $5F + 7S = Z$ defines a straight line with slope $-5/7$, called an **iso-profit line**, along which every production plan yields the same profit Z . Translating this line outward in the direction of increasing profit—up and to the right—increases Z until the line reaches its last point of contact with the feasible region, which occurs at one of the region's corner points (vertices).

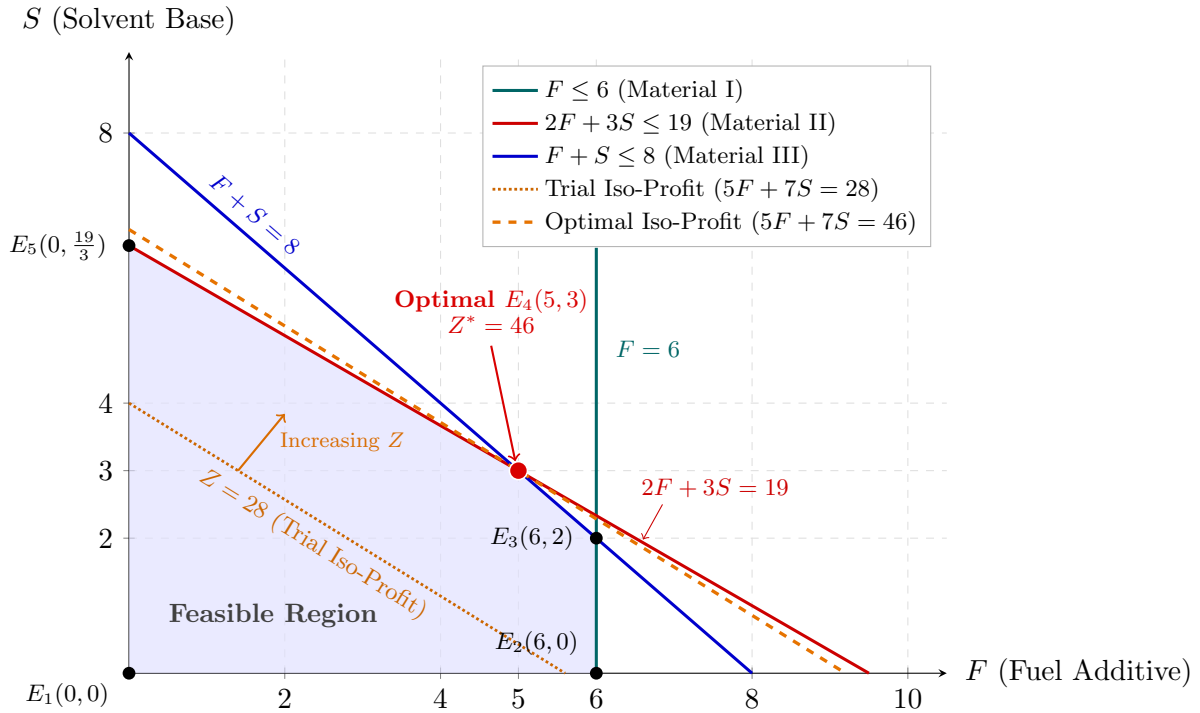


Figure 1: Feasible region, corner points E_1 – E_5 , and iso-profit lines for the RMC, Inc. problem.

As shown in Figure 1, the outward-moving iso-profit line attains the maximum profit at the corner point $E_4 = (5, 3)$, where the boundary lines for Material II and Material III intersect:

$$\begin{cases} 2F + 3S = 19 \\ F + S = 8 \end{cases} \implies (F^*, S^*) = (5, 3), \quad Z^* = 5(5) + 7(3) = \$46.$$

5 Step 4: Managerial Report (Backward Translation)

To complete the modeling cycle, we translate the mathematical optimum $(F^*, S^*) = (5, 3)$ back into a production schedule and resource audit for RMC management:

1. **Optimal Production Plan:** Produce $F^* = 5$ units of fuel additive and $S^* = 3$ units of solvent base.
2. **Maximum Profit Contribution:** This schedule generates a total profit contribution of $Z^* = 5(5) + 7(3) = \$46$.
3. **Resource Utilization Audit:** Substituting $(F^*, S^*) = (5, 3)$ into the three material constraints reveals which resources limit production:
 - **Material I:** Usage is $1(5) + 0(3) = 5$ tons out of 6 tons available, leaving 1 ton of unused capacity.
 - **Material II:** Usage is $2(5) + 3(3) = 19$ tons out of 19 tons available; the available supply is exhausted.
 - **Material III:** Usage is $1(5) + 1(3) = 8$ tons out of 8 tons available; the available supply is exhausted.

6 Extended Topics

Beyond the basic four-step modeling workflow, several structural properties of linear programs govern why corner-point searches work and how high-dimensional models are solved algebraically.

6.1 Extreme Points and the Fundamental Theorem of Linear Programming

Why did the outward-moving iso-profit line in Figure 1 make its last contact with the feasible region at a corner of the polygon?

Because each linear inequality defines a closed half-space, the feasible region is a **convex polygon** (or a **convex polyhedron** in higher dimensions). In a convex set, a point is called an **extreme point** (also called a **vertex** or **corner-point feasible solution**) if it does not lie strictly between any two other points in the set; geometrically in $\mathbb{R}^2$, an extreme point occurs where two constraint boundary lines intersect and all other constraints are satisfied. Because the objective function $Z = 5F + 7S$ is linear, it cannot attain a strict local maximum in the interior of the feasible region or in the middle of an edge without also attaining that maximum at an endpoint of the edge. This geometric principle is the **Fundamental Theorem of Linear Programming**:

*If a linear program has a non-empty, bounded feasible region (or more generally, if it has a finite optimal value over a standard polyhedron), then at least one optimal solution occurs at an **extreme point** of the feasible region.*

This theorem reduces an infinite search over the continuum of feasible plans to a finite check of the extreme points. As shown in Figure 1, the RMC feasible polygon has exactly five extreme points, E_1 through E_5 . Table 2 evaluates the profit $Z = 5F + 7S$ at each extreme point, confirming that $E_4(5, 3)$ yields the highest profit.

Table 2: Enumeration of extreme points for the RMC, Inc. feasible region.

Vertex	Coordinates (F, S)	Active Boundary Equations	Profit $Z = 5F + 7S$
E_1	$(0, 0)$	$F = 0$ and $S = 0$	$5(0) + 7(0) = \$0$
E_2	$(6, 0)$	$F = 6$ and $S = 0$	$5(6) + 7(0) = \$30$
E_3	$(6, 2)$	$F = 6$ and $F + S = 8$	$5(6) + 7(2) = \$44$
E_4	$(5, 3)$	$2F + 3S = 19$ and $F + S = 8$	$5(5) + 7(3) = \$46$ (Optimal)
E_5	$(0, \frac{19}{3})$	$F = 0$ and $2F + 3S = 19$	$5(0) + 7(\frac{19}{3}) = \$44.33$

Notice that not every pairwise intersection of boundary lines produces a feasible extreme point. For instance, the Material I line ($F = 6$) and the Material II line ($2F + 3S = 19$) intersect at $(F, S) = (6, \frac{7}{3})$, but substituting this point into the Material III constraint gives $6 + \frac{7}{3} = \frac{25}{3} > 8$. Because $(6, \frac{7}{3})$ violates the Material III capacity, it lies outside the feasible polygon and is an *infeasible corner point*.

6.2 Standard Form of a Linear Program

Graphical inspection works in two dimensions, but industrial linear programs often involve thousands of variables. To search across extreme points algebraically (for example, via Dantzig's simplex algorithm), we convert inequalities into a uniform system of linear equations known as **standard form**.

A linear program is in **standard form** when it satisfies four structural conditions:

1. **Standardized optimization direction:** The objective function is written in a single canonical direction (here, maximization; a minimization problem $\min c^T x$ is equivalent to $-\max(-c^T x)$).
2. **Equality constraints:** Every structural constraint is a linear equation ($=$) rather than an inequality ($\leq$ or $\geq$).
3. **Non-negative right-hand sides:** Every right-hand-side constant satisfies $b_i \geq 0$. If an equation has $b_i < 0$, multiplying both sides by -1 restores $b_i \geq 0$.
4. **Non-negative variables:** Every decision variable satisfies $x_j \geq 0$. If an original variable $x_j \in \mathbb{R}$ is unrestricted in sign (free), we substitute $x_j = x_j^+ - x_j^-$ with $x_j^+ \geq 0$ and $x_j^- \geq 0$.

6.3 Slack Variables and Surplus Variables

To convert inequality constraints into equations while preserving non-negativity, we introduce auxiliary variables that measure the gap between the left-hand side and the right-hand side:

- **Slack Variables ($\leq$ Constraints):** A “less than or equal to” constraint $\sum_{j=1}^n a_{ij}x_j \leq b_i$ typically represents a resource ceiling. We add a non-negative **slack variable** $s_i \geq 0$ to the

left-hand side:

$$\sum_{j=1}^n a_{ij}x_j + s_i = b_i, \quad s_i \geq 0.$$

Economically, $s_i = b_i - \sum_{j=1}^n a_{ij}x_j$ measures the *unused resource capacity* (idle inventory or machine hours).

- **Surplus Variables ($\geq$ Constraints):** A “greater than or equal to” constraint $\sum_{j=1}^n a_{ij}x_j \geq b_i$ typically represents a minimum requirement, such as a contractual delivery quota or a minimum quality specification. We subtract a non-negative **surplus variable** $e_i \geq 0$ from the left-hand side:

$$\sum_{j=1}^n a_{ij}x_j - e_i = b_i, \quad e_i \geq 0.$$

Economically, $e_i = \sum_{j=1}^n a_{ij}x_j - b_i$ measures the *excess production or over-fulfillment* above the required threshold b_i . For example, if RMC signed a supply contract requiring at least 6 units of combined output ($F + S \geq 6$), we would write $F + S - e_4 = 6$ with $e_4 \geq 0$. At the optimal schedule $(F^*, S^*) = (5, 3)$, the surplus variable equals $e_4^* = 5 + 3 - 6 = 2$ units above the minimum quota.

- **Objective Coefficients of Slack and Surplus Variables:** Because neither unused raw material from existing inventory nor over-fulfillment above a minimum requirement directly generates incremental sales revenue, both slack and surplus variables have a coefficient of 0 in the objective function.

6.4 The RMC Model in Standard Form and Basic Feasible Solutions

Introducing slack variables $s_1, s_2, s_3 \geq 0$ for Materials I, II, and III converts the RMC model into standard form:

$$\begin{aligned} \text{Maximize } Z &= 5F + 7S + 0s_1 + 0s_2 + 0s_3 \\ \text{s.t. } F + s_1 &= 6, \\ 2F + 3S + s_2 &= 19, \\ F + S + s_3 &= 8, \\ F, S, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

This linear system has $m = 3$ equations in $n = 5$ non-negative variables (F, S, s_1, s_2, s_3) . In linear algebra, setting $n - m = 2$ variables (called **non-basic variables**) equal to zero and solving the 3×3 system for the remaining $m = 3$ variables (called **basic variables**) yields a **basic solution**. When all three basic variables are non-negative, the solution is a **basic feasible solution (BFS)**.

Algebra and geometry stand in one-to-one correspondence: *every basic feasible solution of the standard-form equations corresponds to an extreme point of the feasible region*. Setting a variable to zero places the solution on the corresponding boundary line:

- Setting $F = 0$ and $S = 0$ gives $(s_1, s_2, s_3) = (6, 19, 8) \geq 0$, which gives the extreme point $E_1(0, 0)$ at the origin.

- Setting $s_2 = 0$ (Material II boundary) and $s_3 = 0$ (Material III boundary) forces $2F + 3S = 19$ and $F + S = 8$, yielding the optimal extreme point $E_4(5, 3)$ with slack values:

$$\begin{aligned} s_1^* &= 6 - F^* = 6 - 5 = 1 && (1 \text{ ton of Material I remains unused}), \\ s_2^* &= 19 - (2F^* + 3S^*) = 19 - 19 = 0 && (\text{Material II is exhausted}), \\ s_3^* &= 8 - (F^* + S^*) = 8 - 8 = 0 && (\text{Material III is exhausted}). \end{aligned}$$

6.5 Binding, Non-Binding, and Redundant Constraints

Evaluating the slack and surplus variables at an optimal solution classifies constraints into three distinct categories:

- **Binding (Active) Constraint:** A constraint is binding if it holds with equality at the optimal solution ($s_i^* = 0$ or $e_i^* = 0$). In the RMC problem, the Material II and Material III constraints are binding at $(5, 3)$. Geometrically, binding constraints intersect to form the optimal extreme point and block further movement in the profit-improving direction. Economically, they are *bottleneck resources*: acquiring an additional ton of Material II or Material III would shift the boundary outward and increase maximum profit (giving these resources a positive *shadow price*).
- **Non-Binding (Inactive) Constraint:** A constraint is non-binding if it holds as a strict inequality at the optimal solution ($s_i^* > 0$ or $e_i^* > 0$). In the RMC problem, the Material I constraint ($F \leq 6$) is non-binding at $(5, 3)$ because $s_1^* = 1 > 0$. Removing a non-binding constraint leaves the current optimal point $(5, 3)$ unchanged, even though the constraint cuts off part of the feasible region; specifically, without $F \leq 6$, the feasible polygon would extend to $(8, 0)$ instead of being truncated along the segment from $E_2(6, 0)$ to $E_3(6, 2)$.
- **Redundant Constraint:** A constraint is redundant if removing it leaves the *entire feasible region* unchanged because it is already implied by the remaining constraints. For example, if we added the constraint $F \leq 9$ to the RMC model, it would be redundant: since $F + S \leq 8$ and $S \geq 0$ already force $F \leq 8$ for every feasible point, the condition $F \leq 9$ never touches or trims the feasible polygon. A *non-binding* constraint (inactive at the optimum, such as $F \leq 6$) must therefore be distinguished from a *redundant* constraint (inactive everywhere, such as $F \leq 9$).

7 General Model Formulation

The structure developed for RMC extends directly to an arbitrary production problem with n products and m resources.

7.1 Parameters (Exogenous Inputs)

- $j \in \{1, 2, \dots, n\}$: index for products.
- $i \in \{1, 2, \dots, m\}$: index for resources (raw materials).
- c_j : unit profit contribution of product j .
- b_i : available capacity of resource i .
- a_{ij} : units of resource i consumed per unit of product j produced (the *input-output* or *technology* coefficient).

7.2 Decision Variables

- x_j : quantity of product j to produce ($j = 1, 2, \dots, n$).

7.3 Scalar and Summation Forms

1. Expanded Scalar Form:

$$\begin{aligned}
 &\text{Maximize } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
 &\text{s.t. } \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m, \end{aligned} \\
 &x_1, x_2, \dots, x_n \geq 0.
 \end{aligned}$$

2. Summation Notation:

$$\begin{aligned}
 &\text{Maximize } Z = \sum_{j=1}^n c_jx_j \\
 &\text{s.t. } \begin{aligned} \sum_{j=1}^n a_{ij}x_j &\leq b_i \quad \text{for } i = 1, 2, \dots, m, \\ x_j &\geq 0 \quad \text{for } j = 1, 2, \dots, n. \end{aligned}
 \end{aligned}$$

7.4 Matrix Representation

Collecting the scalars into column vectors and a matrix yields a compact linear-algebraic formulation that applies in any dimension.

1. Vectors and Technology Matrix:

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m,$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}.$$

2. **Canonical Inequality Form and Standard Equality Form:** Using the inner product $c^T x = \sum_{j=1}^n c_j x_j$ and componentwise vector inequalities, we write the canonical model and its standard-form counterpart (with slack vector $s \in \mathbb{R}^m$ and identity matrix I_m) as:

Canonical Form

$$\begin{aligned}
 &\text{Maximize } Z = c^T x \\
 &\text{s.t. } \begin{aligned} Ax &\leq b, \\ x &\geq 0 \end{aligned}
 \end{aligned}$$

Standard Form

$$\begin{aligned}
 &\text{Maximize } Z = c^T x + 0^T s \\
 &\text{s.t. } \begin{aligned} Ax + I_m s &= b, \\ x &\geq 0, \quad s \geq 0. \end{aligned}
 \end{aligned}$$

7.5 The RMC Case in Matrix Form

Specializing to the RMC problem with $n = 2$ products ($x_1 = F$, $x_2 = S$) and $m = 3$ raw materials gives:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad c = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \quad (c^T = [5 \quad 7]), \quad b = \begin{bmatrix} 6 \\ 19 \\ 8 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}.$$

In canonical matrix form, the RMC linear program is:

$$\begin{aligned} \text{Maximize } Z &= [5 \quad 7] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{s.t.} \quad &\begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 6 \\ 19 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Appending the slack vector $s = [s_1 \quad s_2 \quad s_3]^T$ and the 3×3 identity matrix I_3 gives the augmented standard-form system:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 19 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \geq 0.$$